NEW YORK UNIVERSITY

Institute for Mathematics and Mechanics

PROGRESS REPORT

On Activities and Work Done Under Contracts

N6ori-201, Task Order 1
Nonr-285 (01)
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Nonr-285 (06)
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Report on activities and work done under Contracts N6ori-201, Task Order No. 1 (supported jointly by the Mathematics Branch of the ONR and the Air Force), Nonr-285(01), Nonr-285(02) (supported by the AEC through the ONR), and Nonr-285(06) (supported by the Mechanics Branch of the ONR), during the period July 1951 - December 1952. In addition, the main part of a report already submitted to the Office of Ordnance Research (Contract DA-30-069-ORD-835) by L. Bers has been included in Part II of this report for the sake of completeness.

This report is divided into two parts. In Part I a general summary of the work and plans of the Institute is given. This summary is not complete; in particular, it does not include classified work done in the Institute. On the other hand, it refers to some activities and aspects that are not directly concerned with the work done under the contracts specified above, but it was thought best to widen the report in this way in order to give a better understanding of the organization, working habits, philosophy, and future plans of the group. In Part II more detailed individual reports are given, as well as a list of published papers, doctoral theses, and lecture notes.

PART I

We have, in general, continued our research program in applied fields of specific interest to the armed forces and to the AEC. We have also continued our program of training combined with research in accord with our basic philosophy that these two functions should never be separated. In addition, although we have carried out certain specific research

projects, we have maintained a reasonable balance between applied and basic research and have made strenuous efforts to preserve a spirit of free enquiry.

It is hardly necessary to say that the shortage of highly trained scientific people persists in our fields of interest as it does in others. We are making every effort to do our share in improving that situation. It is therefore not out of place to mention that we are at present conducting 24 different lecture courses in the Graduate School of Arts and Sciences of New York University which are attended by an average of 25 students in each course. During the period since, and including, June 1951 twenty-three doctor's degrees were awarded by our department; names and thesis titles are given in Part II. As a further educational measure, the younger people working in our Institute (numbering about 25) have all been assigned to one or more of the following working groups, which meet at intervals to discuss the relevant problems, under the leadership of the persons named:

1. Mathematical Physics--Professor Courant
2. Partial Differential Equations--Professor John
3. Quantum Mechanics--Professor Friedrichs
4. Numerical Analysis--Professor Isaacson
5. Spherical Waves--Professor Keller
6. Fluid Dynamics (Mathematical Aspects of Gas Dynamics) --Professor Bers
7. Statistical Mechanics--Professor Grad
8. Water Waves--Professor Peters, Professor Stoker
9. Classical Groups--Professors Friedman, Magnus, Shapiro
10. Number Theory--Professor Shapiro

Our aim in setting up these working groups is as much to promote the education of the young people as it is to further research in concrete fields.

Reports of finished research work have been printed largely in our journal, the "Communications on Pure and Applied Mathematics," a volume of which is issued each year.

In addition, a number of mimeographed reports have been issued. As part of our educational activity we have for years been accustomed to prepare lecture notes of many of the graduate courses in a form suitable for general distribution. This activity has expanded during the past year and a half because of increasing demand from outside, and there are now available lecture notes of thirty different courses.

The preparation of the revised English edition of Courant-Hilbert, "Mathematical Physics," is approaching completion. A book on gravity waves in water by J. J. Stoker will be completed in the course of 1953. A series of five extensive papers by K. O. Friedrichs on the mathematical aspects of field quantum theory has been published in the Communications, and these papers will be collected and supplemented to form a book. A book on partial differential equations by F. John and Peter Lax, and a book by H. Grad on statistical mechanics, are being written.

One of the essential activities of the Institute consists in fostering cooperation with groups and individuals both in this country and abroad. To this end we have had the advantage of visits for varying lengths of time by a considerable number of scientists such as Dr. Schardin (St. Louis in France), Dr. Danel (Grenoble, France), Professor Lighthill (Manchester, England), Professor Fjeldstad (Norway), Professor Heisenberg (Germany), Professor Fichera (Italy), Dr. Richard Meyer (Manchester). Various members of the Institute have visited other research centers, e.g. R. Courant, F. John, J. Keller, and K. O. Friedrichs in Europe last summer (1952). All but the last named also attended the International Congress of Applied Mechanics in Istanbul and gave lectures there, R. Courant one of the invited addresses. J. J. Stoker visited in Europe in the late summer of 1951 as American delegate to an International Symposium on Vibrations in France, and visited for six weeks and gave a course in hydrodynamics at the Applied Mathematics Research Group in

Stanford University during the summer of 1952. Peter Lax spent a month last summer, pant of the time at Stanford, part of the time at Los Alamos. L. Nirenberg spent the year 1951-52 on leave of absence in Europe, where he visited at length in Zurich (at the Technische Hochschule) and paid shorter visits to research centers in Germany, France, and Italy.

Finally, before outlining in a summary way the activities in the various scientific fields, some mention of changes in the University and of new enterprises affecting our Institute should be made. New York University has had during the past year a new Chancellor, Dr. Henry T. Heald (formerly of Illinois Institute of Technology). In addition, a number of new administrative posts have been created, such as that of Executive Vice-Chancellor, Dr. David D. Henry (formerly President of Wayne University). One of the recent acts of the new administration has been the creation of an Institute of Mathematical Sciences, under the leadership of R. Courant, which will comprise the present Institute for Mathematics and Mechanics, the present research group working on problems in electromagnetic waves under Professor Morris Kline, and a new group which has been formed to carry out a contract with the AEC to operate a UNIVAC digital computer at New York University. A building has been purchased by the University which will put all three of these groups under one roof. In effect, the University is supporting the activities of our, and related, groups more vigorously and effectively than ever before. A development in the field of statistics and probability is planned.

Part I of this report concludes with a brief summary of the scientific topics in which the Institute has been active. This is supplemented in Part II by reports prepared by a number of individuals in the Institute, and which should be consulted for more detailed statements about the work done on specific problems.

A. Mathematical analysis in general

1. Partial differential equations. This subject continues to be one of the more engrossing for a considerable group, including Friedrichs, John, Bers, Lax, Douglis, Nirenberg, and a number of their students and collaborators. The completed work in this field includes the following: (1) A proof of the identity of weak and strong elliptic differential operators of a rather general character and with non-constant coefficients (Friedrichs). (2) Papers on weak solutions of elliptic partial differential equations and on the fundamental solution of linear analytic partial differential equations of elliptic type (F. John). (3) A new proof of Carleman's unique continuation theorem, and the development of a theory of pseudo-analytic functions and integrals on closed Riemann surfaces (Bers). (4) A study furnishing bounds for the size of the domain of regularity for a hyperbolic system, and studies on equations of the type $U_{tt} = -AU$ with U a vector and A a matrix (Peter Lax).

2. Numerical methods. This subject has become increasingly interesting to more and more people in the group--a tendency which is likely to increase because of the presence of a large group operating the UNIVAC. In particular, the problem of obtaining numerical solutions when the basic problem involves partial differential equations has been much to the fore. Finite difference schemes and their convergence for a single first order equation $u_t = au_x + bu$ with $u(x,0) = \varphi(x)$ and a and b variable have been studied by Peter Lax. A new scheme for calculating shocks by finite differences has been devised by Peter Lax which seems to have quite considerable advantages over other methods and to be of great practical value. A paper on the solution of nonlinear hyperbolic differential equations by finite differences (R. Courant, E. Isaacson, M. Rees) has been published in the Communications. A paper by F. John on integration of

parabolic equations by difference methods has also been published in the Communications. Doctor's theses in this field by Milton Rose and L. Nemerever have been completed under the supervision of E. Isaacson.

3. Functional analysis. This is still another field in which interest is on the increase. Among the work done was: (1) A proof that elliptic operators of second order over a compact domain of n-space have a complete set of eigenfunctions under the first, second, or third boundary condition (Lax and Berkowitz). (2) A discussion of the behavior of the solution of a problem in elasticity involving the biharmonic operator was carried through using abstract methods (Lax and Berg). (3) A summary and exposition of the work of Friedrichs and others concerning the problem of criteria for continuous or point spectra has been worked out by J. Berkowitz. (4) Friedrichs and Shapiro have been working on a general theory of integration of functionals.

4. Number theory. H. Shapiro has continued his work in analytic number theory and supervised the thesis by Forman in this field. (See Part II for a more extended summary of this work.)

B. Mathematical physics

One of the major tasks of the Institute during the past years has been the study of the mathematical basis of quantum field theory by a group (including Corson, Zumino, Moses) under the leadership of K. O. Friedrichs. The results of these studies (as has been mentioned earlier) have appeared in a series of articles in the Communications, and they will be collected and published as a book. The work of this group continues, but it is not expected that it will be pursued quite as intensively in the future as it has during the past few years.

Here, as with the other subdivisions, a good deal of overlapping with other fields occurs. For example, many of

the types of problems to be mentioned subsequently involve considerations in the domain of wave propagation, and some of the results obtained by J. Keller and his collaborators, by the group working on water wave problems, and by the groups concerned with spherical waves in gas dynamics and with problems in meteorology have made contributions with some general interest in the theory of wave propagation. There have also been some contributions in potential theory by Peter Lax, and by various people working with water wave problems.

The seminar in mathematical physics (under the supervision of R. Courant) has been studying the theory of Hill's equation with particular reference to the proposed new design of a cyclotron at Brookhaven and will continue with a study of the Liapounoff theory.

C. Fluid mechanics

This field has always been a major preoccupation at the Institute, and it will continue to be so.

In compressible gas dynamics L. Bers and his collaborators (Agmon, Protter, Berg) have been interested in existence and uniqueness questions for subsonic flow past given profiles, with and without wakes, and existence proofs have been given, by functional analysis methods (Schauder-Leray, essentially), in both cases. The second case--that of cavitation flows--comprises the doctor's thesis of Paul Berg. Certain aspects of the problem of transonic flow have been studied by the same group with the addition of C. Morawetz and C. Gardner. I. Kolodner and C. Morawetz have obtained a short and simple proof (without assuming analyticity) of Friedrichs's theorem on the non-existence of the so-called limiting line. It is suspected that a flow past a profile with a local supersonic region is mathematically impossible in general, and various aspects of this non-existence statement have been studied by C. Gardner and C. Morawetz. Uniqueness theorems with reference to linear differential equations of mixed type (the

Tricomi equation, especially) have been studied (Protter, Agmon, Morawetz).

The propagation of spherical waves in compressible media continues to be studied intensively. G. B. Whitham has developed and published in the Communications a method of treating weak spherical shocks. This theory has been extended and used to study the propagation of spherical waves in a star with inclusion of the forces due to gravitational attraction; this work has application also to wave propagation in shallow water of variable depth. C. Morawetz has studied the propagation of very strong spherical shocks. Attempts are now being made to deal with the difficult in-between region in which the spherical shock is neither very strong nor very weak.

The work on water waves continues. A. S. Peters and J. J. Stoker are engaged in studying the motion of a ship under very general conditions: the ship is treated as a floating rigid body with its six degrees of freedom, and the motion of the ship under the action of the propeller thrust and a given sea-way is to be determined in terms of parameters fixing the shape of the ship's hull. This program is feasible, it turns out, and solutions even in some rather complicated cases can be given explicitly. At worst, an integral equation must be solved. S. Ciolkowski is studying the problem of determining the point spectrum in the case of three-dimensional waves over sloping beaches by extending the methods used by A. S. Peters to obtain the continuous spectrum. Two doctor's theses have been completed recently in this field. One is by H. Rubin, who gives an existence proof in the case of progressing waves in the presence of a finite dock. The other is by A. Finkelstein, who gives rather general uniqueness theorems for the transient and unsteady motions, including cases in which rigid bodies act as obstacles to the wave motion. Certain unsteady motions have been studied by J. J. Stoker with the object of clarifying the

conditions at ∞ for steady states (the so-called Sommerfeld conditions) by studying the limits as $t \to \infty$ of properly formulated problems for unsteady motions. A report on the theory of floating breakwaters in shallow water and its application to breakwaters in the form of rigid bodies and flexible beams has been written by B. Fleishman, J. J. Stoker, and L. Weliczker, and it will be issued in a few weeks. Nonlinear problems in the theory of water waves have also been studied. F. John has devised a means of producing explicit unsteady as well as steady flows of liquids under gravity and having a free surface. A. Troesch has been studying the problem of the breaking of waves by means of approximate theories which are more accurate than the classical nonlinear shallow water theory. K. O. Friedrichs has devised a new approach to the old classical and famous problem of proving the existence of the solitary wave; the details of this method, which looks very promising, are being worked out by Professor D. H. Hyers, who is visiting at the Institute while on leave from the University of Southern California. L. Nirenberg is taking up once more similar problems concerning steady waves due to flows over obstacles, which he had carried rather far before going on leave of absence last year.

D. Nonlinear vibrations

Some work has been done in this field, but the interest in it at present is slight. A doctor's thesis in this field has been completed by B. Fleishman. This thesis, which used largely methods devised by F. Ficken for this type of problem, was concerned with proving the existence of a time-periodic solution in a nonlinear system having infinitely many degrees of freedom with provision for large viscous damping. F. Ficken and J. J. Stoker collaborated in working on the same type of problem without damping; this work was discussed by the second-named at an International Symposium held on the Ile de Porquerrolles in France. J. Keller has

considered the problem of the bowing of a violin string by taking into account the self-excited character of the excitation, and has been able to carry out explicit solutions in a variety of cases.

E. Statistical mechanics

A group consisting of H. Grad, Marian Rose, P. Mostov, R. Goldberg, and others, under the leadership of the first-named, has intensified the work in this field. Completed work includes: (1) Determination of the profile of a plane shock wave using the thirteen moment equations, (2) statistical mechanics of systems with many integrals, (3) derivation of the Boltzmann equation, (4) plane Couette flow. Work is nearly completed on the problems of molecular chaos and sound dispersion. Work continues on problems concerning diffusion, proof of a general H-theorem, theory of dense gases and liquids, specific gas dynamical flows for molecules with internal structure (P. Mostov), slow flow around a sphere with arbitrary mean free path (R. Goldberg), cylindrical Couette flow for thirteen moments (Marian Rose). The work in this field continues to promise good returns.

F. Shaped charges

Aside from classified reports (by Hudson and Gardner), the work on the problems of this field by G. E. Hudson included a study of Mach reflection of strong shocks at a rigid wall. The work under contract in this field will terminate shortly.

G. Meteorology

The efforts in this field have all been directed toward the study of the large-scale perturbations of the prevailing westerly currents in middle latitudes. The first step in this direction was a derivation by Keller and Ting of

simplified equations from the exact hydrodynamical equations through a formal development with respect to a parameter in a manner reminiscent of the way one obtains the nonlinear shallow water theory of gravity waves. Equations similar to those derived by Charney and Thompson (and which serve as a basis for numerical prediction of pressure changes in the atmosphere) were obtained, but unfortunately, as was shown by Lowell, these equations lead only to rather uninteresting solutions. Lowell has since made strenuous efforts to obtain developments of a similar character in other forms, but the problem seems to be difficult because the development in all probability has asymptotic rather than convergent character and its form is very hard to guess. Isaacson and Lowell have studied the nonlinear differential equation of Charney (used at Princeton as a basis for numerical prediction) from a mathematical point of view and have found, for example, various different ways in which it is reasonable to formulate the boundary and initial data. Stoker has developed a theory for discussing the motion of cold and warm fronts, on the basis of a series of fairly plausible physical assumptions, which is not bound to the assumption of small perturbations on the discontinuity surface between the wedge of cold air at the ground and the overlying warm air. The resulting nonlinear partial differential equations have then been treated approximately in two different ways. One way, devised by Whitham, leads rather easily to a qualitative discussion of the dynamics of frontal motions that is in excellent accord with the observations, but this approach has the disadvantage that it cannot be carried out (as far as can be seen now, that is) quantitatively because of a peculiar difficulty at cold fronts. Another way, devised by Stoker, consists in making still further simplifying assumptions in order to obtain a system of differential equations in two independent variables only: the time and one space variable. A system of four first order nonlinear partial differential equations is

obtained which can be solved numerically by using finite differences applied to the characteristics. The calculations are very tedious, but they have been encouraging so far in that the qualitative phenomena seem to be correctly obtained. Work in this direction will be continued.

H. Nonlinear elasticity

Two doctoral theses have been completed recently in this field. M. Yanowitch has discussed the stability of certain nonlinear bent states of the thin circular elastic plate. One case is that of the symmetrical buckled state of the circular plate under pressure in the plane of the plate, and the other is the symmetrical bent state due to pressure applied normally to the face of the plate. In both cases instabilities arising through perturbations from the symmetrical bent state were found under appropriate boundary conditions and at sufficiently high values of the applied pressures.

The thesis of B. Altshuler was concerned with the nonlinear buckling of the spherical thin shell under constant external pressure, and the main object was the study of asymptotic developments capable of yielding the buckling into small inward dimples which is observed in experiments.

PART II

Lipman Bers

Subsonic Flow: Existence and Uniqueness Theorems

The problem of finding a subsonic potential flow past a profile P which is horizontal at infinity amounts to a boundary value problem for a nonlinear partial differential equation. More than a hundred papers were written on this subject, but with a few exceptions, all of these deal either with the construction of examples or with approximate procedures of doubtful validity.

In 1934 Frankl and Keldysh proved the existence and uniqueness of a solution for sufficiently small values of the free stream Mach number M_∞. A few years ago I proved the existence of a solution with arbitrarily given maximum local Mach number $M_{max} < 1$, but only for the so-called "$\gamma = -1$ case," that is for the equation of minimal surfaces. In 1951 Shiffman proved a general existence theorem for values of M_{max} arbitrarily close to 1. Shiffman uses variational methods and assumes P to be smooth. He prescribes M_∞ and the value Γ of the circulation, and proves uniqueness only within the class of functions for which a certain Dirichlet integral is finite. From the aerodynamical point of view, however, the interesting case is that of a profile P with a sharp trailing edge, Γ being not prescribed but determined by the Kutta-Joukowski condition. A complete solution of this problem has been achieved during the work on this project.

We assume that P is a simple closed curve which possesses at all points, save perhaps one, z_T, a tangent whose inclination as function of arc-length satisfies a Hölder condition. At z_T the two one-sided tangents exist and form the angle α,

$0 < \alpha \leq \pi$. It is required to find a potential gas flow past P such that (i) on P the velocity vector is tangent to P and satisfies a Hölder condition, (ii) at z_T the speed q vanishes (Kutta-Joukowski condition) (if $\alpha < \pi$ it suffices to require that q should be finite at z_T), (iii) as $(x,y) \to \infty$ the y-component of the velocity vector approaches 0, the x-component approaches a given positive number q_∞.

Our main result asserts that there exists a number $\hat{q} > 0$ such that for $0 < q_\infty < \hat{q}$ the problem has a unique solution depending continuously on q_∞. This solution is everywhere subsonic, that is $M_{max} < 1$. M_{max} is a continuous function of q_∞. As $q_\infty \to 0$, $M_{max} \to 0$; as $q_\infty \to \hat{q}$, $M_{max} \to 1$. There are no stagnation points outside P. If $q \neq 0$ on P except at z_T, then q vanishes at z_T of order $1 + (\alpha/\pi)$. Otherwise q vanishes at z_T of order α/π, at some other point at P of order 1, and $q \neq 0$ elsewhere. If P is symmetric with respect to the x-axis, the M_{max} is a monotonic function of q_∞.

The proof of this theorem is based on the Schauder-Leray method, as was the previous work on the case $\gamma = -1$. The same approach yields other existence and uniqueness theorems. Thus one may prescribe M_∞ and require that $\Gamma = 0$ (this amounts to finding the direction of no lift), or one may prescribe M_∞ and the location of the second stagnation point, etc. Also, one may assume that at z_T the profile has a cusp ($\alpha = 0$), and that there is a finite number of intruding corners.

A complete presentation of these results will be given in a separate paper.

The techniques developed for treating a gas flow past a profile can be used also for discussing cavitation flows. In a thesis written by Paul Berg these techniques have been used to solve the Helmholtz wake problem for a compressible fluid.

Subsonic Flows: Mathematical Background

Analytic functions and conformal mapping are the proper tools for investigating incompressible flows, and also compressible flows in the Chaplygin approximation ("$\gamma = -1$"). The corresponding tools for subsonic gas dynamics are two generalizations of complex function theory: the theory of quasi-conformal mappings (Grötsch, Teichmüller, Lavrentyeff, Ahlfors) and that of pseudo-analytic functions (Položii, Bers). In the course of the work on this contract some new results and new applications of these theories have been obtained.

(A) A general property of variable density flows. By a variable density flow is meant a solution of the elliptic equation $(\rho\phi_x)_x + (\rho\phi_y)_y = 0$ where the unknown function ϕ is interpreted as the velocity potential of a flow with a given variable density $\rho(x,y)$. Using some simple theorems on pseudo-analytic functions and quasi-conformal mappings it can be shown that every such flow is topologically equivalent to an incompressible flow. Thus one can give an a priori description of the stream-line pattern of a gas flow satisfying given boundary conditions. These results apply to both subsonic and transonic flows. A complete statement will be given in the paper mentioned above.

(B) Boundary value problems for the equation $\Delta\phi + \alpha(x,y)\phi_x + \beta(x,y)\phi_y = 0$. The theory of these problems is classical, but the usual treatment assumes α, β to be regular on the closure of the domain considered. The proof of the theorems described in the preceding section required a theory of boundary value problems for the case when α, β become infinite at the boundary. Such a theory has been developed by the use of pseudo-analytic functions. Results which have gas-dynamical applications are included in the paper mentioned above. Other results may be published separately at a later date.

(C) Carleman's theorem. This theorem asserts that solutions of a system of differential equations of the form $u_x - v_y = \alpha u + \beta v$, $u_y + v_x = \gamma u + \delta v$ have isolated zeros. It is basic for the theory of pseudo-analytic functions and its applications to gas dynamics. A strengthened version and a new proof of this theorem have been obtained.

(D) Pseudo-analytic functions on closed Riemann surfaces. This work is relatively remote from applications but seems to be of independent interest. It consists of extending the basic theorems on Abelian integrals and the Riemann-Roch theorem to the case of pseudo-analytic functions and differentials. The results have been reported to the recent Princeton Conference on Riemann surfaces and a summary will appear in the Proceedings of this conference. A complete presentation will be published later.

Transonic Flows: The Non-existence Problem

Some time ago many aerodynamicists believed that a potential gas flow past a profile breaks down only after a sizable locally supersonic region has been developed. An attempt was made to explain this break-down by the appearance of a so-called "limiting line" in the hodograph plane of the flow. This latter hypothesis was disproved in an important paper by Friedrichs. Friedrichs' proof, however, is rather complicated and is based on not quite natural analyticity assumptions. In connection with the work of the seminar I. Kolodner and C. Morawetz obtained a very short and simple proof of Friedrichs' theorem. Their paper will appear in the Communications on Pure and Applied Mathematics.

It is now generally believed that a potential flow past a profile with a local supersonic region is mathematically impossible except for very special cases. More precisely, the following non-existence theorem has been conjectured: if

a transonic potential flow is possible for a profile P and free stream Mach number M_∞, then no such flow with the same value of M_∞ exists for a profile P' arbitrarily close to P. In a series of papers Frankl, Busemann and Guderley tried to give plausibility arguments supporting this conjecture, and to reduce it to a uniqueness theorem for linear partial differential equations of mixed type. Their reductions always depended on considering "infinitely close" flows and this introduced a new element of uncertainty. Recently C. Gardner gave a very simple argument, avoiding the consideration of infinitesimal variations, which also reduces the proof of the non-existence theorem to a linear uniqueness theorem. Unfortunately neither the theorem conjectured by Gardner nor the more special uniqueness theorems conjectured by Frankl and Guderley have as yet been proved. Gardner's argument is given in his seminar talk.

A rigorous but very special non-existence theorem is due to Nikolski and Taganov. Let there be given a potential gas flow past an arc C and let C_o be a subarc of C such that $M > 1$ along C_o and $M = 1$ at the end-points of C_o. The theorem states that the supersonic region extends to infinity if C_o contains a straight segment. Recently C. Morawetz obtained a considerably stronger version of this theorem. She showed that the same conclusion holds if C_o contains a point with vanishing curvature. This work is not yet prepared for publication.

Transonic Flows: Mathematical Background

The theory of transonic flows depends on the theory of partial differential equations of *mixed type*. The simplest such equation is Tricomi's equation

$$y\,\psi_{xx} + \psi_{yy} = 0. \tag{1}$$

The stream-function of a gas flow considered as a function of

the velocity components satisfies an equation which can be reduced to the Chaplygin form

$$K(y)\,\psi_{xx} + \psi_{yy} = 0 \tag{2}$$

with

$$K(y) = y[1 + O(1)], \qquad y \rightarrow 0.$$

After the appearance of Tricomi's pioneering paper (1923) many authors investigated equations (1) and (2) but we are still very far from a general theory.

For equation (1) Tricomi succeeded in finding a well-set boundary value problem. A problem involving transonic gas jets led Frankl to consider the same problem for Chaplygin's equation. He proved a uniqueness theorem under a restrictive condition on the size of the domain. A less restrictive uniqueness theorem has been obtained by Protter. Recently Protter also attacked the existence theorem using methods developed by Gellersted. This approach seems successful, but the work is not yet ready for publication. An investigation of the same problem by different methods is being carried out presently by Agmon. The work is as yet in too early a stage to be reported on.

The non-existence theorem for transonic flows discussed above requires for its proof a uniqueness theorem for a boundary value problem generalizing that of Tricomi. No proof of such a theorem has yet been found, except for an isolated result by Protter, which is too special to be stated here.

It is hoped that during the coming year a major part of the work on the project will be devoted to the theory of equations of mixed type. Only major progress in this direction can lead to a real understanding of the mathematical aspects of transonic flows.

Avron Douglis

Most of A. Douglis' work this year has been concerned with linear, hyperbolic equations of second order with variable coefficients. An elementary means was developed for showing, without making use of the fundamental solutions or of Riesz integrals, that the solution of an initial-value problem for such an equation is characterized by an integral relation of Volterra type. This integral relation was then applied to the study of the behavior of solutions of Cauchy problems. These studies, which are still in progress, are concerned chiefly with characterizing the domain of dependence and with investigating the effect on the solution of discontinuities in the initial data.

The above procedures were modified in such a way as to apply to certain problems in which both Cauchy data on a space-like initial manifold and boundary data on a time-like manifold are prescribed. Now an attempt is being made to generalize these preliminary results and thus to develop a new comprehensive theory of composite initial- and boundary-value problems. Such a new theory would not, indeed, take the place of the existing theories of these problems, but would be of advantage, it is believed, in studying certain aspects of the behavior of solutions.

Lastly, some preliminary attempts have been made to adapt these ideas to the treatment of hyperbolic equations of higher order. There is as yet, however, nothing conclusive to report.

K. O. Friedrichs

My primary occupation consisted in continuing work on the quantum theory of fields which I had started about two years before. I finished parts IV and V of the series of papers "Mathematical Aspects of the Quantum Theory of Fields."

In part IV a general treatment of the occupation representation is given. Such a representation has been employed so far by physicists only in the case in which the field is enclosed in a box, so that the possible energy eigenstates of a single particle form a discrete set, Φ_r, $r = 1, 2, \cdots$. The state of the field is then described by the amplitude $f(\nu_1, \nu_2, \nu_3, \cdots)$ of the probability that ν_r particles are in the state Φ_r. The restriction that the variable r on which the "occupation number" ν depends is discrete, is removed in part IV. Occupation numbers $\nu = r(s)$ are introduced which depend on a variable s with which a general measure differential dm(s) is associated. The probability amplitude is then a functional $f(\nu)$ of $\nu(s)$. For these functionals a quadratic unit form is introduced, written in the form

$$(f,f) = \sum_{\nu} |f(\nu)|^2 \prod_s \frac{(dm(s))^{\nu(s)}}{\nu(s)!} .$$

This symbolic notation proves useful in evaluating various significant quantities associated with the field.

A different class of occupation functionals $\varphi(\nu)$ is introduced, associated with a unit form written in the form

$$(\varphi,\varphi) = \sum_{\nu} |\varphi(\nu)|^2 \prod_s \frac{(dm(s))^{\nu(s)}}{\nu(s)!} e^{-dm(s)} .$$

It turns out that in case $\int dm(s) = \infty$ a field which has such occupation functionals as probability amplitudes is essentially different from fields of the kind described before.

The name "myriotic" is introduced for fields of this new type. The most remarkable property of myriotic fields is that they possess no vacuum state and that the expected value of the number of particles composing it is always infinite.

These notions are first developed for boson fields and finally carried over to fermion fields.

In part V a general theory of linear transformations Y of annihilation and creation operators $A^{\pm}(x)$ into a new pair $B^{\pm}(x) = Y A^{\pm}(x)$ is developed.

The elements of the matrix Y are operators acting on functions of x. A special class of matrices P is introduced such that the transformation Y = exp P produces operators $B^{\pm} = Y A^{\pm}$ which satisfy the same commutation rules as the operators $A^{\pm}$. The problem now is to find a unitary transformation T acting on the state of the field such that $B^{\pm} = T^{-1} A^{\pm} T$. It is shown that such a "canonical transformation" can be given in the form T = exp [P] when [P] is a quadratic form in the A^{+} and A^{-} whose coefficients are essentially those of the matrix P. The important problem now is to order the transformation T with respect to annihilation and creation operators, specifically to write T in the "E-ordered" form

$$T = \exp [P_+] \exp [P_o] \exp [P_-],$$

in which P_+ and P_- depend only on the A_+ and A_- respectively, and in which P_o commutes with the number of particles. These matrices $P_{\pm}$, P_o can be obtained simply by decomposing the matrix exp P in the form

$$\exp P = \exp P_+ \exp P_o \exp P_-.$$

The derivation of this fact is based on a fundamental theorem by Baker and Hausdorff (1906).

The results are employed to analyze various types of "infinities" or "divergencies" occurring in the quantum theory of elastic media and in the theory of vacuum polarization.

As a small contribution to the mathematical aspects of the quantum theory I gave a unified treatment for the time-dependent and time-independent scattering process. The time-independent treatment is given with the aid of Heisenberg's S-matrix from which the limit as $t \to \infty$ of the probability of scattering in any direction can be derived. The time-dependent treatment of the scattering of particles which arrive with a sharply defined momentum leads to a probability which increases linearly in time. In the unified treatment a slightly unsharp initial state is assumed. Then the probability is approximately linear in time only over a certain time interval, whose extent depends on the unsharpness, while it approaches a finite limit as $t \to \infty$.

Another contribution to the mathematical aspects of quantum theory is a paper on the General Adiabatic Theorem. It is assumed that the Hamiltonian H depends on the time t in the form $H = H_{t/t_1}$. The process of switching on the difference $H_1 - H_o$ will thus be completed in the time t_1. It is desired to find an asymptotic expansion of the solution $\psi(t,t_1)$ of the Schroedinger equation $i\nabla_t\psi = H_\psi$ for large values of t_1. The first term of this expansion gives the Adiabatic Theorem. The proper evaluation of this term depends on the nature of the spectrum of H. In the case of a discrete spectrum the second term of the expansion is determined. In the case of a continuous spectrum the relationship of the Adiabatic Theorem to the "irrelevancy of the adiabatic hypothesis" is established. In a simple problem of field quantum theory the Schroedinger equation can be solved exactly. The correctness of the first term derived from the theory developed can thus be verified.

A contribution to the "pure" theory of partial differential equations is a paper on the Identity of Weak and Strong Solutions of Linear Elliptic Differential Equations. A general differential operator L of order 2r is considered. Its leading term is written in the form $\tilde{D}^r a D^r$, in which D stands for the gradient, $-\tilde{D}$ for the divergence, while the matrix a, which depends on the point x in the domain, satisfies an appropriate positive definiteness condition. The operators $\tilde{D}^r$ are assumed to be applicable in the strong sense with respect to quadratic integrability in closed subdomains. The operators $\tilde{D}^r$ are assumed applicable in the weak sense. It is proved that the solution u of the equation $Lu = f$ admits the strong operator D^{k+2r}, if the right member f admits the strong operator D^k. The two main tools consist first in integral inequalities of the type used for similar purposes in a paper by Courant, Lewy, and the author in 1928, second, by approximation by smoothing integral operators, the "mollifiers." The above result is used to prove that the solution u possesses derivatives in the strict sense of order s if the right member f possesses strong derivatives up to an order $k > s - 2r + m/2$.

E. M. Corson

Dr. Corson has carried out studies as follows:

In the theory of quantized fields, the investigation of the interrelation of the Feynman-Dyson Schwinger formalisms and the underlying physical meaning of the renormalization techniques. A principal difficulty of interpretation seems to arise through the appearance, and requirement, of two quite distinct sets of density-tensors (charge-current, energy-momentum), the one renormalized and formally playing the usual role, the other being required to obtain the usual three-dimensionally integrated total field quantities. This situation seems to imply a completely different connection whose physical meaning is, as yet, not clear.

In the theory of crystal statistics, the investigation of the two-dimensional binary lattice--first with a combinatorial

approach collateral to the usual group-theoretical treatment as given by Onsager-Kauffman, et al. A particular mode of combinatorial solution has been anticipated, during the course of this study, and further consideration has been directed towards the possibility of extending this technique to the three-dimensional case. As yet no further solutions have been obtained, and we are attempting to verify Kac's adumbration that his result, in principle, suggests that the general case is not solvable.

Bruno Zumino

Dr. Zumino has in particular been concerned with the following topics:

The relation with the variational Lagrangian formulation of Schwinger has been established. Work is in progress making use of these results, hoping to obtain a consistent formulation of quantum electrodynamics, involving only renormalized, observable quantities. The object is to write down the equations for the observable quantities in closed form, and to be able to investigate their exact solutions, giving in this way an answer to the fundamental problem of the convergence of these solutions after renormalization.

A partial problem has been particularly taken into consideration: the polarization of the vacuum in a given external non-quantized electromagnetic field. Exact expressions for the vacuum induced current charge densities can be obtained using a method of Friedrichs, applied by him to a problem having a similar mathematical structure. An infinite renormalization of the charge is necessary also in the exact solution, in order to make results finite. Work on these lines is still in progress.

On a more preliminary level some features of the theory of non-interacting quantized fields have been investigated. The correct definitions of the density of number of particles in coordinate space in a relativistic theory and the connection with the one particle theory and with Pryce's investigation on the center of mass have been established.

Some classical relativistic nonlinear equations have been investigated in connection with Heisenberg's theory of multiple creation of mesons. Relations have been found between the regions of hyperbolicity of the equations and the regions of reality of the associated energy momentum tensor.

Harold Grad and collaborators

Completed and published within the last year:

(1) Profile of a Plane Shock Wave--the solution is given for the thirteen moment equations and is compared with the results of Becker, Thomas, Wang Chang, and Mott-Smith--Communications, Vol. V, No. 3, Aug. 1952.

(2) Statistical Mechanics of Systems with Many Integrals --contains a more rational presentation of conventional statistical mechanics, shedding new light on the Gibbs Paradox, and extends the results to systems of many integrals--Communications, Vol. V, No. 4, and J. of Phys. Chem., Dec. 1952.

Completed and being prepared for publication:

(3) Derivation of the Boltzmann Equation--more rigorously and in more generality than has been done previously--this clarifies the significance of irreversibility and ergodicity.

(4) Plane Couette Flow--solution using thirteen moment equations valid for the whole range zero to infinite mean free path--awaiting completion of cylindrical Couette flow [see Marian Rose, below].

Almost complete:

(5) Molecular Chaos--the problem is formulated mathematically and a proof given in the general non-equilibrium case.

(6) Sound Dispersion--work completed last year is being supplemented by boundary conditions which will make experimental verification easier.

Work continuing on:

(7) Diffusion--a variety of problems are accessible.

(8) Spherical polynomials and spherical harmonics in n-dimensions.

Significant results can be expected in:

(9) Proof of a general H-theorem.

(10) Theory of dense gases and liquids.

(11) Specific gas dynamical flows for molecules with internal structure--[see P. Mostov, below].

R. Goldberg--obtained locally Maxwellian solutions of Boltzmann equation with gravity; computed thirteen moment equations in general and cylindrical coordinates; working on slow flow around sphere with arbitrary mean free path.

Marian Rose--cylindrical Couette flow for thirteen moments; major difficulties overcome.

P. Mostov--obtained 17-moment equations for Boltzmann equation for rough spheres; to continue with more general molecular models and solution of specific problems.

Philip M. Mostov

In detail: Grad's moment method of solution of the Boltzmann equation was successfully extended by Mr. Mostov to a molecular model with rotational degrees of freedom--the perfectly rough, perfectly elastic spherical molecule. Employing the rotational energy per unit mass and the rotational heat flow vector, in addition to the thirteen moments used by Grad, a system of seventeen equations in seventeen moments was derived. The equations are capable of handling phenomena whose interesting behavior depends on rapid variations in time or space, or on the rotational and translational degrees of freedom having different effective temperatures. The results are more general than those of Pidduck's and Kohler's treatments by the Chapman-Enskog method, and include their results as special cases. The equations are general enough in character to afford a more penetrating insight into the conditions for which the definitions of the usual macroscopic coefficients are valid, and the conditions for which various time constants are important.

Values were obtained for the shear viscosity, volume viscosity and generalized heat conduction coefficients (it is possible to have the translational and rotational temperatures different even in the steady state); and for the time constants associated with the stresses, the translational and rotational heat flows and the rotational-translational

temperature difference. Several important special applications which help define the various macroscopic coefficients and time constants (and their limitations) and which help illustrate the use of the moment equations were examined.

The results of this investigation were presented at the Cambridge meeting of the American Physical Society, and are being readied for publication very shortly.

Several fruitful generalizations, modifications and applications contemplated are:

1. Other types of molecular interactions.

2. Consideration of the effect of various parameters of the investigation in ranges not usually treated.

3. The effect of the rotational degrees of freedom on the thickness of the single shock profile, and on

4. The dispersion and absorption of sound waves, especially at high frequencies or low densities.

5. The generalization of Grad's methods to quantum mechanical models.

6. The generalization of the equations to include the effects of radiation to and from molecules, and their application to problems in the upper atmosphere, planetary atmospheres and interstellar space.

E. Isaacson

(1) Spherical Shock Waves--checked work of C. Morawetz.

(a) For underwater shocks the first perturbation term (about Primakoff, Taylor solution) has been explicitly obtained. The perturbation series is probably an asymptotic one and to evaluate fully the effect of the first term it will probably be necessary to analyze some experimental data. We are making efforts to locate this information.

(b) For shocks in air, the method of determining the first perturbation is worked out. But a great deal of numerical integration would be involved if practical answers were desired. We plan to complete (a) before continuing with (b).

(c) Work of Whitham on shock propagation at large distances remains to be integrated with above efforts.

(2) Work on finite difference methods

(a) Completion of Courant, Isaacson, Rees--On the Solution of Nonlinear Hyperbolic Differential Equations by Finite Differences.

(b) Assisted Milton Rose, who wrote Ph.D. thesis on convergence of finite difference methods for second order hyperbolic equations. (He also extends work of Bers to parabolic equations; and has some general remarks on mathematical basis of relaxation procedure for solving F.D. equations of elliptic problems.)

(c) Helped L. Nemerever, who wrote Ph.D. thesis on F.D. methods for solving axially symmetric nozzle flow problems. The coefficients of the hyperbolic F.D.E. are singular at the axis. He establishes convergence for the characteristic initial value problem (based on Y. W. Chen's theorem on the existence of solutions).

(3) Work in meteorology--helping J. J. Stoker

(a) Initial attack on the determination of the proper mathematical problem for a peculiar third order partial differential equation (an approximation used by Charney and others for study of long waves in the upper atmosphere).

$$\begin{cases} \frac{D}{Dt}(\Delta \psi) = 0, & \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \\ & \psi_x = u \\ & \psi_y = v. \end{cases}$$

I had hoped to work with Sherman Lowell on this problem.

(b) Guiding G. Booth in his handling of study of movement of cold fronts for J. J. Stoker.

(4) Work in water waves--helping J. J. Stoker

(a) Guiding L. Weliczker in work on floating breakwaters and in study of the error in shallow water approximation.

(5) Computations.

Future outlook

(1) Complete appraisal of work of C. Morawetz on spherical waves and combine with Whitham's work. (J. Keller will probably push this now.)

(2) Continue analysis of meteorological problems, especially (3a).

(3) Consider use of finite difference methods for various versions of the partial differential equations of meteorology.

(4) Analysis of finite difference methods for symmetric hyperbolic systems--to follow work of Friedrichs--and perhaps specialize to problems of Maxwell's equations.

(5) Consider investigation of promising finite difference methods tried by P. Lax, which avoid separate consideration of shocks and other discontinuities.

(6) Investigate finite difference methods for parabolic equations with undetermined boundaries.

F. John

Numerical analysis

Completed manuscript started at Institute for Numerical Analysis, establishing convergence of difference schemes for solutions of linear and quasi-linear parabolic difference equations for given initial values in an unbounded interval. Proof that difference scheme converges and gives the correct solution for Riemann-integrable initial data, provided the v. NEUMANN stability criterion is satisfied. (Appeared in Communications 5 (1952), pp. 155-211.)--RITTERMAN works on thesis to establish similar results for equations in a bounded interval, with boundary conditions prescribed at the endpoints. Emphasized is the case often occurring in applications to diffusion and other problems, where the prescribed initial and boundary data are not perfectly consistent initially, and hence singularities of the solution are present.

Incompressible flows with free boundaries

The stimulus to this investigation was the consideration of the flow of the liner in a shaped charge. A formal representation for the most general analytic transient 2-dimensional flow of a heavy liquid with a free boundary was derived, in terms of the general solution of a simple linear parabolic equation. This work is related to a remark of GARDNER and HUDSON on constructing flows by means of solutions of parabolic equations, and to a representation given by H. LEWY for steady 2-dimensional flows with a free boundary. (Reported at the International Congress on Theoretical and Applied Mechanics at Istanbul.)--Independently of this a special simple 3-dimensional flow was constructed, in which the free surface of the liquid is an ellipsoid varying with time. (To appear in Revue d'Hydraulogique.)--SKINNER is working on

extending these results and to represent the general 3-dimensional transient flow with rotational symmetry of a heavy liquid with a free boundary. This would complement results of LEWY and GARABEDIAN, who found an expression for the general steady flows of this type. A number of explicit examples of 2-dimensional flows were computed and plotted by the staff of the Institute, according to the general formulae given.

Water waves

FINKELSTEIN wrote a thesis on infinitesimal waves of a liquid of constant or infinite depth. He constructs first of all the Green's function corresponding to waves originating from a momentary disturbance at a point on or below the surface of the liquid. With the help of this function it is possible to analyze the behavior at infinity of any transient waves originating from disturbances in a bounded region. A consequence is a uniqueness theorem more general than any derived previously, stating that the motion of the liquid is uniquely determined by initial conditions and by boundary conditions on any obstacles in the liquid having a prescribed motion. The Green's function also permits to reduce determination of the liquid motion to an integral equation.

Hyperbolic differential equations

A thesis was written by GARDNER about problems arising for the wave equation, in which the solution depends only on the data in a portion of the domain of dependence one would have expected ordinarily. This research was stimulated by aerodynamical problems involving flow past a thin airplane wing, but also brings up a number of basic questions for the wave equation, which only recently have attracted attention of a number of mathematicians, e.g. of MARCEL RIESZ.

A study of the Cauchy problem for a hyperbolic equation on a time-like manifold has been begun. Necessary and

sufficient conditions for Cauchy data to be admissible on a time-like plane have been determined for the wave equation. The conditions are of an intricate nature, involving analytic continuation of functions. Necessary conditions have been found for the solvability of the Cauchy problem on time-like manifolds for the most general types of differential equations. (Not yet completed.)

Elliptic equations

An elementary proof for the existence of derivatives of weak solutions of a linear elliptic differential equation was given. The method used employs "spherical means," a technique used previously by the author. Results obtained parallel those given by K. FRIEDRICHS and F. BROWDER, who proceed differently, but the results obtained here are somewhat weaker than the ones of those authors. (Results to appear in the Communications, and a shorter version also in the Proceedings of the Conference on Partial Differential Equations at Arden House.)

J. B. Keller

I. Research published in journals

1. Reflection and Diffraction of Pulses by Wedges and Corners (with A. Blank), Comm. on Pure and App. Math., June 1951, Vol. IV, No. 2

2. Comment on "Channels of Communication in Small Groups," American Sociological Review, Vol. 16, No. 6, Dec. 1951, pp. 842-843

3. Parallel Reflection of Light by Plane Mirrors, Quarterly of App. Math. (in press)

4. Diffraction of a Shock or an Electromagnetic Pulse by a Right-angled Wedge, Jour. of App. Physics, Nov. 1952, Vol. 23, No. 11, pp. 1267-8

5. Reflection of Water Waves due to Surface Tension and Floating Matter (with E. Goldstein), Trans. of Am. Geophysical Union (in press)

6. Finite Amplitude Sound Waves, Jour. of Acoustical Soc. of Am. (in press)

7. On Bohm's Interpretation of Quantum Mechanics, Phys. Review (in press)

8. Reflection and Transmission Coefficients for Water Waves on Floating Ice (with M. Weitz), Comm. on Pure and App. Math. (in press)

9. Variational Treatment of Water Wave Reflection (abstract only), Gravity Waves Symposium Volume, 1952.

II. Research published in project reports

A. IMM-NYU

1. Instability of Liquid Surfaces and the Formation of Drops (with I. Kolodner) . . . . June 1952

2. Summary of Six Memos on Underwater Explosion of Atomic Bombs . . . . . . June 1952

3. Decay of Spherical Sound Pulses due to Viscosity and Heat Conduction . . Aug. 1952

4. Decay of Droplets by Evaporation and Growth by Condensation (with I. Kolodner, P. Ritger) 1952

5. Thin Unsteady Heavy Jets (with M. Weitz) 1952

6. Planar, Cylindrical and Spherical Flows of a Compressible Fluid 1952

7. Geometrical Acoustics I: The Theory of Weak Shocks 1952

8. Underwater Explosion Bubbles I. The Effect of Compressibility of the Water (with I. Kolodner) . . . . in press

9. Underwater Explosion Bubbles II. The Effects of Gravity on Bubble Shape (with I. Kolodner) . . . . in press

B. Washington Square College Mathematics Research Group

1. On Systems of Linear Ordinary Differential Equations (with H. Keller) . . . . July 1951

2. Parallel Reflection of Light by Plane Mirrors . . . . Oct. 1951

*3. On Slightly Non-circular Wave-guides (with H. Keller) 1952

*4. Diffraction of a Shock or an Electromagnetic Pulse by a Right-angled Wedge 1952

5. On the Scope of the Image Method . . in press

C. Unpublished

1. Factorization of Matrices

2. The Field on a Caustic (with I. Kay)

3. On the Direction of the Force of Sliding Friction

*Also published in Journals.

4. Derivation of the Meteorological Equations (with L. Ting) [to be revised]

5. Bowing of Violin Strings

III. <u>In progress</u>

A. Asymptotic Solution of Partial Differential Equations (with I. Kay, E. Bauer)

1. Diffraction thru Apertures

2. Derivation of Geometrical Optical Theory of Water Waves

3. Derivation of the Bohr-Sommerfeld Quantum Theory

4. Calculation of Field on Caustics and Foci

5. Diffraction at an Interface

6. Analysis of Diffracted Rays and Wavefronts by Geometrical Optics

B. Formation and Behavior of Aerosols (with I. Kolodner, P. Ritger, M. Jordan)

1. Exact solution for a single drop evaporating

2. Exact and approximate solution for a drop evaporating in a spherical container, and for many drops

3. Formation of drops from a plane viscous sheet of fluid and from a non-viscous spherical sheet

C. Scattering of Electromagnetic Waves (with H. Keller)

1. Reflection from rough surfaces

2. Inversion of matrices in multiple scattering problems

D. Spherical Waves (I. Kolodner, G. Whitham, C. Morawetz)

1. Weak Spherical Shocks and Kirkwood-Bethe Theory--by G. Whitham (in press)

2. Perturbation of the Strong Shock Solution--by C. Morawetz (in press)

3. Reflection and diffraction of shocks at the ground, and in heated layers

Ignace I. Kolodner

1. Research on Underwater Explosions

The new method developed (in summer 1951) to handle the problem of motion of underwater gas globes without artificial restrictions on their shape has been extended to properly account for the influence of a bottom at finite depth and of the free water boundary. The theoretical part is now complete and Miss Reisman has done some computation for a special case suggested by A. B. Arons to compare our predictions with his experimental data.

A second phase of this piece of research which concerns itself with the influence of obstacles is now well advanced, but not completed. Work on it has been interrupted for a while.

An oral report on the first part was given in an address at the Conference on Undex, sponsored by the Bureau of Ships (December 1951), and the subject was discussed again in a seminar lecture at the Brooklyn Polytechnic Institute (March 1952).

Preparation for the publication of these results has been delayed until the second phase is completed. It is intended to present in a future publication (with Joseph B. Keller) a full revision of existing theories on the subject.

2. Research on Aerosols and Sprays

The initial stage of study of drop formation may be now considered complete. In addition to the initial report, two other reports on the subject were published:

(1) (with J. B. Keller) Instability of Liquid Surfaces and the Formation of Drops (June 1952, IMM-NYU 182);

(2) (with J. B. Keller and P. Ritger) Decay of Drop by Evaporation, and Growth by Condensation (July 1952, IMM-NYU 183).

The first of the above discusses the production of drops by breakup of an accelerated plane layer of liquid. Mr. Whitham took over this part and now studies the case of an accelerated spherical layer. The second report gives an approximation to the solution of the evaporating droplet problem. Mr. Ritger is now applying the methods used in this report to the case involving a finite number of drops.

A justification of the method used in the latter report is now completed and is in the write-up stage. This includes a proof of existence and uniqueness of solutions of the heat flow equation with moving boundary. A problem of similar structure was considered two years ago by G. Evans. The present method of proof has the following advantages over the techniques used by Evans:

(1) The proof is constructive and is easily generalized to more complicated cases.

(2) It leads to a sharp majorant and minorant for the unknown boundary function, thus eliminating the need for numerical computations in all practical cases.

Peter D. Lax

Spectral Theory (with J. Berkowitz)

Proved that elliptic operators of second order over a compact domain of n space have a complete set of eigenfunctions, under the first, second or third boundary condition. The eigenfunctions satisfy the boundary conditions in a generalized sense only, and generalized eigenfunctions have to be admitted (if the index of an eigenvalue is greater than one).

Pursuant to the work of Browder (and Gårding) we were able to extend this result to higher order elliptic operators using the same method, which is to analyze the resolvent as a meromorphic function in the complex plane. Similar results were obtained by Browder, using different methods. In finding the requisite estimates we relied on the halfboundedness of elliptic operators (under the first boundary condition).

Hyperbolic Differential Equations

The width of the domain of existence of an initial value problem $U(x,o) = \phi(x)$ for a hyperbolic system $U_t + AU_x + B = 0$ depends on the C_1 norm of ϕ, i.e. on upper bounds for the first derivatives of the initial values of all the unknowns U. I found that in some exceptional cases the range of t depends only on upper bounds for n - 1 of the unknowns (there is a corresponding simplification in the manner of dependence of the solution on the initial values). Important examples of such exceptional systems are the equations of one-dimensional time-dependent flow with variable entropy, and two-dimensional steady rotational supersonic flow. In the first case, e.g., the range of t for which the solution will exist as a continuous flow depends on upper bounds for the initial values of u_x and p_x but <u>not</u> on the derivative of the second thermodynamic

variable. From this result I was able to deduce that flows with contact discontinuities are limits of continuous flows.

The exceptional case is characterized by this condition: let d denote one of the eigenvalues of A; $d = d(x,t,U)$ depends on the variables since A does. Denote by $E(x,t,U)$ the corresponding right eigenvector. The system is exceptional if the gradient of d with respect to U is orthogonal to E, and in this way it is possible to obtain discontinuous solutions as limits of continuous ones; the line of discontinuity will be a characteristic curve with slope d.

Hyperbolic Equations in more than one space variable

The abstract theorem which states that if a transformation T is bounded with respect to a Banach norm, and symmetric with respect to a Hilbert norm, then it is bounded with respect to the Hilbert norm (provided that the scalar product is continuous in the Banach norm) throws new light on the role of the $\mathcal{L}_2$ norm for the initial value problem. One can show namely with its aid that if having a finite norm with respect to any Banach norm is a continuable (or persistent) property of the initial values, then having a finite $\mathcal{L}_2$ is also a persistent property. The reasoning applies not only to symmetric hyperbolic systems but to general hyperbolic systems, investigated by Petrowsky, as well.

Second Order Equations

I have shown that the initial value problem $U = \phi$, $U_t = \psi$ for the equation

$$U_{tt} = -AU$$

can be solved provided that A is of the form $\tilde{A} + N$, where $\tilde{A}$ is a self-adjoint operator bounded from below, $(u,\tilde{A}u) \geq (u,u)$, and N is bounded with respect to the norm induced by $\tilde{A}$, i.e. $\|Nu\|^2 \leq \text{const.}\ (u,\tilde{A}u)$. This theorem includes many known

concrete results; its proof relies on the Hille-Yosida theorem, operating with the norm $\|(\phi,\psi)\|^2 = (\phi,\tilde{A}\phi) + (\psi,\psi)$.

<u>Fourth Order Elliptic Operators</u> (with Paul Berg)

The alternative for the solubility of the first boundary value problem (u and $\frac{\partial u}{\partial n}$ prescribed) was demonstrated for fourth order operators which are the products of second order operators L and M, having the same principal part. The method consists of reducing the statement of the alternative to this <u>geometric</u> statement about the nullspaces $\mathcal{L}$ and $\mathcal{M}$ of the operators L and M: <u>The projection of</u> $\mathcal{L}$ <u>into</u> $\mathcal{M}$ (under the $\mathcal{L}_2$ scalar product) <u>covers all but a finite dimensional component of</u> $\mathcal{M}$, <u>whose dimension is equal to the dimension of those elements in</u> $\mathcal{L}$ <u>which are orthogonal to</u> $\mathcal{L}$.

The reduction is brought about by interpreting the differential equation and the boundary condition in their integral (weak) form.

<u>Biharmonic Equation</u> (with Paul Berg)

In connection with a problem of the theory of elasticity, Friedrichs conjectured that all solutions of the biharmonic equation in a halfstrip $0 \leq x \leq 1$, $0 \leq y$ whose boundary data u and $\frac{\partial u}{\partial n}$ are zero on the sides $x = 0$, $x = 1$, $0 \leq y$, decay exponentially. We proved this conjecture with the aid of the concepts and methods of the theory of semigroups, the semigroup considered being formed by the transformations $T(y)$ mapping the data u, u_y at $y = 0$ into their values at y. The starting point of these investigations was R. Phillips' work on the spectral theory of semigroups; in the course of events we found that for semigroups whose elements $T(y)$ are <u>completely continuous</u> the results of Phillips--the spectral mapping theorem--are particularly simple (this class of semigroups is included among the ones studied by Phillips). In particular we find the largest value of λ such that every

solution of the above description of the biharmonic equation in the strip is less than const. $e^{-\lambda y}$, namely the minimum of the real part of those roots of the transcendental equation $\sin \lambda = \pm \lambda$ whose real part is positive.

Numerical Integration of Partial Differential Equations

Finite difference schemes for the initial value problem for a single first order equation: $U_t = au_x + bu$, $U(x,o) = \phi(x)$ were investigated; attention was restricted to those schemes for which $u(x,t+h)$ was expressed as linear combination of values of u at one step back, t. It was shown that schemes that are stable in the sense of von Neumann are convergent, even if the coefficients a and b are variable. The estimates nec ssary for the proof were made in the maximum norm; the method of the proof bears a strong resemblance to Fritz John's method for parabolic equations.

Shock Calculations

In work performed for the Los Alamos Scientific Laboratory a new scheme was devised for calculating shocks. The method resembles an earlier one proposed by von Neumann (later modified by Richtmyer and von Neumann) in that it is a straightforward numerical integration scheme. Its new features are: (i) using the hydrodynamic equations as they appear as conservation laws, (ii) using a forward difference scheme.

Experimental calculations performed on the equation $u_t + uu_x = 0$ and the hydrodynamic equations at the Los Alamos Scientific Laboratory indicate that the method works.

Potential Theory

A simple proof is given for the existence of Green's function for domains with smooth boundaries with the aid of the Hahn-Banach theorem (published in the Proceedings of the American Mathematical Society).

S. C. Lowell

(1) A major endeavor was to investigate the possibility of deriving the "equations of large-scale dynamical meteorology" from the full hydrodynamical equations for a baroclinic, adiabatic atmosphere on a rotating earth by means of a parameter expansion in terms of the Rossby number. The Rossby number, which is defined as $U/\Omega L$ where U is a characteristic flow velocity, Ω is the basic rotational velocity and L is a characteristic length scale, apparently is the most important dimensionless physical parameter characterizing large-scale atmospheric waves in middle latitudes. Although a much deeper understanding of the basic mathematical structure of meteorology was obtained, only limited success can be claimed in the attempt to derive an analog for meteorology of the shallow water theory for incompressible fluids. It appears that the equations currently being used by dynamical meteorologists contain terms of different order in any possible Rossby number expansion. Further research on this topic is, however, worth carrying out and it will be continued during the current year.

(2) The study of atmospheric discontinuity surfaces, such as "fronts" and "tropopauses," has also received considerable attention. These surfaces, like shock fronts in gas dynamics, seem to be fundamental in atmospheric processes, yet their role is not clearly understood. Only stationary frontal surfaces have been discussed in any detailed manner. Hence I have been investigating the differential geometry of accelerated discontinuity surfaces and the linkage which they bring about between the warm and cold air masses surrounding them. This work, which is quite general in its approach, will combine with that of Stoker, Whitham and Yanowitch, which involves many specific assumptions but permits a deeper

analysis of the nonlinear effects at fronts, to give a well rounded understanding of frontal phenomena.

(3) The Air Force Cambridge Research Laboratories have been testing numerical weather prediction techniques. I was able to assist them in one phase of this work with advice on analytical and numerical procedures which are now being used.

Cathleen S. Morawetz

1. Spherical Shocks

It has been known that the problem of finding the flow behind a spherical expanding shock of infinite strength and constant energy can be solved in terms of solutions of ordinary differential equations in the case of polytropic gases (Taylor) and explicitly for water ($\nu = 7$) (Primakoff). Treating the pressure ahead of the shock as a small parameter, I have been able to find the first order terms in the expansion of the flow quantities in terms of solutions of ordinary differential equations and explicitly for $\nu = 7$. At present I am trying to check the results for water against experiments, but so far I have not succeeded in finding experimental results for sufficiently strong shocks. I have also studied detonations in a similar way, treating the energy of formation as a small parameter. In addition I have been correcting my earlier work on contracting shock waves.

2. Transonic Flow

I have shown

(1) A limiting line cannot appear in continuous mixed flows depending continuously on a parameter--either the Mach number at infinity or representing a changing profile.

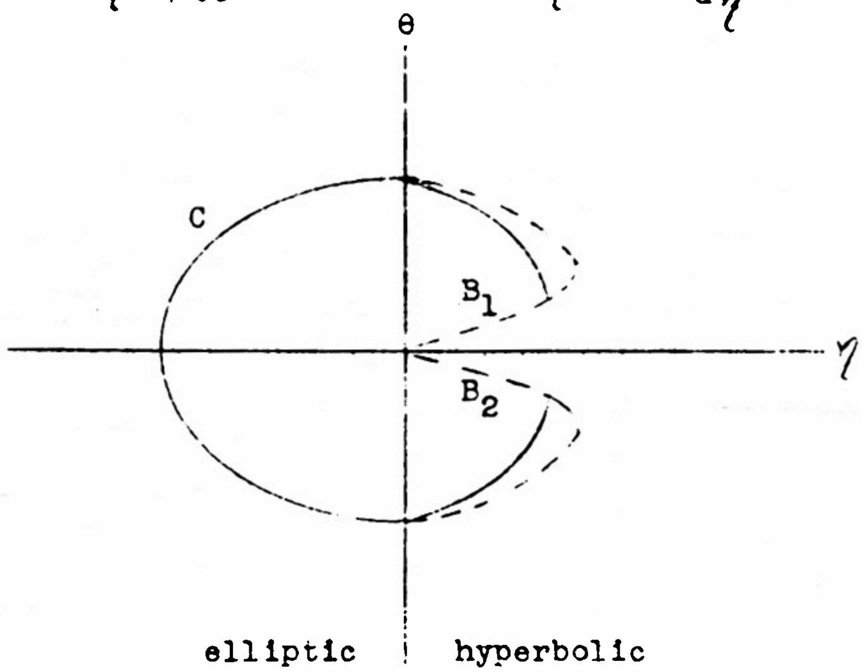
(2) Continuous mixed flow past a profile on which the curvature vanishes somewhere in the supersonic region cannot exist if the curvature of the streamlines is bounded.

(3) If $F(\psi, y) - x = 0$ is an implicit solution of $\psi_{yy} - K(y)\psi_{xx} = 0$ then F satisfies the Monge-Ampère equation,

$$F_{\psi\psi} F_{yy} - F^2_{\psi y} = -K(y).$$

(4) $H = \oint G(\theta)\psi(y,x+\theta)d\theta$ are general solutions of $H_{yy} - K(y)H_{xx} = 0$ if ψ is a particular solution of $\psi_{yy} - K(y)\psi_{xx} = 0$ and $G(\theta)$ is analytic. I have tried to use these solutions to construct profiles and corresponding mixed flows but without success, since I need some singular solution of $\psi_{yy} - K(y)\psi_{xx} = 0$ to start with. This was essentially the method of Tomatika and Tamada for a special choice of K(y).

(5) Uniqueness theorem for Tricomi gap problem. $\psi \equiv 0$ if $\psi_{\eta\eta} - K(\eta)\psi_{\theta\theta} = 0$, $K \gtrless 0$ for $\eta \gtrless 0$, $\frac{dK}{d\eta} \geq 0$ for



$\eta \leq 0$. $\psi = 0$ on C under the following assumptions:

(a) ψ has continuous derivatives in $\bar{D}$ where $\bar{D}$ is the closed region bounded by C and the two characteristics B_1 and B_2.

(b) For $\eta \leq 0$, $\frac{dy}{d\theta} \gtrless 0$ for $\theta \gtrless 0$.

(c) For $\eta > 0$, $\left|\frac{dy}{d\theta}\right| \leq \frac{1}{\sqrt{K}}$ on C.

A. S. Peters

Work was done on the problem of waves on the surface of a body of water bounded by a vertical sided cove or a wedge-shaped region where diffraction phenomena can occur. The problem is in some sense a generalization of the Sommerfeld radiation problem. The mathematical analysis has been completed, but it needs to be revised before publication.

Under the assumption that the motion of a thin ship is caused by its propeller thrust, it was found that the problem could be solved by a perturbation method when the course of the ship is normal to a sinusoidal sea. Expressions, involving wave resistance integrals, were found for the pitching angle, the heave and the surge. Since the analysis and results clarify some misconceptions, they should be of interest to naval people. We would like now to solve the problem in the case where the course of the ship is oblique to the wave fronts. This involves the solution of an integral equation and other aspects which are now under study.

Harold N. Shapiro

I. Publications submitted or appeared

(a) On the Changes of Sign of a Certain Error Function (with P. Erdos), Canadian Journal of Math.

(b) On Iterates of Arithmetic Functions and the Sequence of Primes (submitted to Pacific Journal).

(c) On Distribution of Square-free Integers in Small Intervals (with R. Bellman), Preliminary report to appear in Proc. Nat. Acad. of Science.

(d) The Existence of Distribution Functions for Arithmetic Error Functions (in preparation) (with P. Erdos).

II. Project work

(a) Worked with Professor Friedrichs on his ideas concerning the construction of a "suitable" integration operator in Hilbert space.

(b) Worked with Mr. Forman on abstract prime number theorems, a portion of which work constitutes Mr. Forman's thesis for the Ph.D. degree.

J. J. Stoker

The problems in which J. J. Stoker and a group collaborating with him have been most interested during the past year and a half concerned: (1) gravity waves in water, (2) elasticity, (3) meteorology, (4) nonlinear vibrations. A summary of the work in each follows.

1. Water waves

A. S. Peters and J. J. Stoker have been working on the theory of the waves created by a moving ship under the most general hypotheses which still permit a linear theory. The ship is assumed to be a floating rigid body (with six degrees of freedom) moving under the action of a prescribed propeller thrust in an arbitrary sea-way. The theory is obtained by a formal development with respect to a parameter which is essentially the breadth-length ratio of the ship; a linear theory results when only the lowest order term in such a development is retained. This procedure yields the same theory as that derived by Michell, and used so extensively by Havelock, when one takes the same simple special case, i.e. the case of a hull held fixed in space while the water streams past, and all disturbances are assumed to die out at ∞. The procedure used by us clarifies some points even in this simple case, and permits at least a formally consistent theory in much more complicated cases. (It might be added that we failed to get a consistent theory when we tried to operate without a formal development.) The resulting theory is complicated, but solutions are feasible. In some fairly complicated cases even explicit solutions are possible, for example the case of steady motion of a ship under a constant propeller thrust in a straight course against waves moving at right angles to the course; in this case the trim, rise of the center of gravity, forward speed, and pitching amplitude can be

obtained explicitly although it is necessary to evaluate a number of integrals of the type of the Michell integral. When rolling motion occurs it appears that an integral equation must first be solved before the parameters fixing the motion of the ship can be obtained.

A. S. Peters has completed the solution of wave diffraction problems when the obstacles are of the form of vertical wedges of any angle; in particular, in the case of a barrier in the form of an infinite half-plane, the result is a new method of solution of the classical Sommerfeld diffraction problem. In this same general area S. Ciolkowski is working on the problem of finding the point spectrum for waves on sloping beaches by generalizing the method of Peters for the continuous spectrum. J. J. Stoker has worked out the solution to the problem of the waves created in a uniform stream of velocity U and finite depth h when a disturbance is created at $t = 0$ and maintained for all $t > 0$. As $t \to \infty$ a steady state exists and can be found, except when $U^2 = gh$, when the disturbance velocities tend to become infinite everywhere. This approach makes understandable the rather paradoxical results in the literature which arise when the steady state is treated by imposing radiation conditions at ∞ of the Sommerfeld type (which are not needed in treating the unsteady motion).

A report by B. Fleishman, J. J. Stoker, and L. Weliczker on floating breakwaters in shallow water has been completed. A theory, along the lines of the linear long wave-shallow water theory, has been devised (by Stoker) that is general enough to include practically any type of breakwater. The theory is then applied (by Fleishman) to the case of floating rigid bodies, and (by Weliczker) to floating breakwaters in the form of structures with bending flexibility (i.e. beams). In both cases the reflection and transmission coefficients were determined for progressing waves with normal incidence. The results for floating beams were calculated using design

data furnished by J. Carr of the California Institute of Technology, who has conducted experiments with the object of testing the practical feasibility of such structures to provide breakwaters--for landing operations, say. The calculations, which were extremely tedious and complicated for the case of the beam, led to results that had both positive and negative features but which indicated the direction in which the design should be changed for improved performance. L. Weliczker has also studied the accuracy of the approximate theory in two ways: by comparing with exact results in a special case that can be worked out explicitly, and by extending the approximate theory to the terms of next higher order (essentially in the depth-wave length ratio). The latter study is still in progress.

A doctor's thesis has been completed (under the supervision of F. John) by A. Finkelstein in which uniqueness theorems for unsteady motions in unbounded bodies of water are derived. Included are cases in which bounded rigid bodies are immersed in the water. The principal tool used was a time-dependent Green's function, which was determined explicitly and which had such properties at ∞ as to permit showing the existence of the energy integral. Another doctor's thesis (under the supervision of Stoker and Friedrichs) has been completed by H. Rubin on the problem of progressing waves in the presence of a finite dock. Many unsuccessful attempts to solve this problem explicitly have been made. Rubin does not solve the problem explicitly, but obtains rather an existence theorem. His approach is to formulate an appropriate minimum problem and to solve it by using direct methods of the calculus of variations. Here, it is essential not to work with the velocity potential ϕ (since the energy integral does not exist), but rather with the function $\psi = \phi_y - \phi$ (the operator that occurs in the free surface condition) and to obtain a minimum problem for ψ.

All of the above researches refer to linear theories and thus to waves of small amplitude. Work is also in progress on problems in which the approximation implied in using linear theories is not sufficient. One phase of this work is being carried out by A. Troesch, who is working on the problem of breaking of waves and allied problems by using higher order nonlinear shallow water theories, somewhat along the lines of approximate theories proposed long ago by Boussinesq. The object is to try to find out when and why breaking occurs, and what essential factors condition it. The problem has mathematical features of interest since it leads to nonlinear differential equations of unconventional types which ought to be studied from various points of view--including numerical analysis by finite differences, or otherwise, for example. K. O. Friedrichs has invented an iteration scheme for attacking the famous old problem of the solitary wave which it is hoped will converge to yield the existence of such a wave. The idea is, essentially, to start with the approximate solution furnished by the nonlinear shallow water theory and iterate; the small parameter with respect to which the development takes place is then not the amplitude but rather the ratio of a significant curvature of the free surface to the undisturbed depth of the water. The details are being carried out by D. H. Hyers, who is on leave from the University of Southern California and is visiting our Institute.

During the past eight months J. J. Stoker has been occupied principally with writing a book covering the whole theory of gravity waves in water. The lengthiest and most difficult part of the book is finished (i.e. the part concerned with waves of small amplitude as treated by potential theory). The book should be completed in the course of 1953.

2. Meteorology

Some progress has been made in this field since the last report, but the number of people interested in the problems

has decreased considerably. Our interest has been confined to problems relating to large-scale perturbations in the motion of the prevailing westerlies in middle latitudes. We began by trying to derive approximating differential equations from the exact hydrodynamical equations by means of a formal development with respect to an appropriately chosen parameter. As stated in the previous report, such a development (of a type similar to the nonlinear shallow water theory for gravity waves) was obtained by Keller and Ting. The resulting differential equations were much like those derived by Charney and Thompson on the basis of many ad hoc assumptions. Unfortunately these equations, while correct, were shown by Lowell to yield only results of a rather uninteresting character. Lowell has made strenuous efforts to carry out a formal development in other ways, but it seems to be very difficult to guess the right form for the development, and this is perhaps not to be wondered at since the development doubtlessly has asymptotic rather than convergent character and thus might just as well proceed in fractional powers of the parameter, or in logarithmic terms, for example. Isaacson and Lowell have given some study to the nonlinear differential equation of Charney which serves as a basis for numerical prediction at Princeton, and have uncovered some interesting facts concerning the mathematical theory of the equation.

J. J. Stoker has undertaken the study of a quite different problem, i.e. the problem of the motion of cold and warm fronts. Up to now only linear perturbations of the discontinuity surface between a wedge of cold air and the overlying warm air appear to have been studied. Although such studies have proved to be quite valuable--for example, in determining stable and unstable motions with respect to wave length of the perturbations--it is clear from a glance at any weather map that the actual motion of the fronts develops amplitudes so large and departs so widely from sinusoidal form that a linear theory cannot possibly furnish a good approximation to

these unsteady motions. Stoker has derived a nonlinear theory based on the exact hydrodynamical equations which is reminiscent of the nonlinear shallow water theory in the sense that the vertical coordinate is eliminated through the use of the hydrostatic pressure relation in the same way as was done by Freeman for certain purely two-dimensional motions in the atmosphere. The result is a set of differential equations which is still unmanageable from a computational point of view, principally because there are three independent variables--the time and two space variables. From this point on two different approaches to the solution of the resulting equations have been devised. One of these, by Whitham, involves essentially the exact solution of the equations in each vertical plane parallel to the front, so that hyperbolic equations in two variables only must be integrated, the second space variable playing the role of a parameter; afterwards the dependence on the latter variable is determined by integrating a first order partial differential equation. This approach gives quite good qualitative results, which are in accord with the observations. It has the disadvantage that even a numerical integration is made impossible because of a strange difficulty at any cold front: the situation here corresponds to what would be the propagation of a shock into a vacuum or, better, to the propagation of a bore in shallow water down the dry bed of a stream, and these motions are mathematically impossible to deal with unless provision for a resistance force, or turbulence, or both, is made in some way--and we seem to have no good way to do such things. A different approach to the integration of the approximate differential equations has been devised by Stoker. The basic idea is to get rid of one of the two remaining space variables by making assumptions which have the effect of yielding a "shallow water theory" for a horizontal direction, and thus one thinks of the motion of the fronts as a phenomenon something like the breaking of water waves, only that for fronts

the breaking (corresponding to the occlusion process) takes place in the horizontal rather than the vertical plane. The result is a set of four first order nonlinear partial differential equations in two independent and four dependent variables which are of hyperbolic type. Numerical solutions by finite differences are being carried out by G. Booth, but the calculations are extremely tedious and only meager results are available. However, the qualitative character of the motion seems to be correctly obtained--for example, the cold front steepens relative to the warm front when the initial motion of the originally stationary front (assumed to run in the east-west direction) is a bulge toward the north. These approaches to the problem of a dynamical treatment of the motion of fronts have perhaps one virtue at least: they indicate the lengths to which it is necessary to go to get a theory capable of yielding even qualitatively the observed phenomena. A report on this work by Stoker and Whitham (which we plan to continue) is being written.

3. Elasticity

Two doctor's theses have been completed recently in this field. Both are concerned with nonlinear problems in which the nonlinearity arises because finite displacements are permitted, but linear relations between stress and strain are assumed to hold. The thesis by M. Yanowitch is a study of the stability of equilibrium of two different types of bent states of equilibrium of the thin circular plate. In both cases the bent state in question is symmetrical with respect to the center, and is furnished by appropriate solutions of the nonlinear Föppl-von Kármán equations. In one case the bent state of equilibrium arises through buckling of the plate under thrust applied in the plane of the plate and at its edge (obtained by Friedrichs and Stoker); in the other the bent state arises through pressure applied normally to the face of the plate (obtained by Bromberg). The stability

of both types of equilibrium with respect to perturbations not symmetrical to the center of the plate has been studied in detail by Yanowitch. For the buckled states the result is that the symmetrical state becomes unstable under sufficiently high pressure at the edge (because of a high circumferential compression near the edge) if the edge of the plate is embedded in lateral springs, no matter how stiff--in other words, what might be called a "second buckling" with circumferential wrinkles seems likely to occur. (Friedrichs and Stoker failed to obtain such a second buckling because they assumed the edge of the plate to be held rigid.) In the case of the plate under normal pressure it is shown that the symmetrical bent state becomes unstable at a certain sufficiently high value of the normal pressure provided that the edge of the plate is permitted freedom to move in the radial direction: the instability with respect to unsymmetrical perturbations again results because of the development of circumferential compression in a ring toward the edge of the plate.

A second thesis in nonlinear elasticity was completed by B. Altshuler under the supervision of K. O. Friedrichs. It dealt with the nonlinear buckling of the thin spherical shell under uniform external pressure. The object of the thesis was to deal with the observed nonlinear buckling into small inward dimples by using appropriate asymptotic developments involving a boundary layer treatment. The problem appears to be one of great difficulty, since even to get a formal asymptotic development was far from simple. Such a development was obtained, but the calculations needed to obtain numerical results of a type suitable for checking the observations have not been carried out because of their complexity.

4. Nonlinear vibrations

In this field interest was confined to the study of a nonlinear system having infinitely many degrees of freedom,

with the object of extending the Poincaré theory of the existence of time-periodic solutions in the neighborhood of a given periodic solution. The nonlinear system chosen was the linear vibrating string embedded in lateral springs whose stiffness is a nonlinear function of the lateral displacement. This leads to the classical wave equation in one space variable and the time with a nonlinear term in the dependent variable. The problem of proving the existence of a periodic solution in the neighborhood of a solution of the linearized equation was attacked by F. Ficken (who was visiting the Institute while on leave from the University of Tennessee) by the methods of functional analysis. Unfortunately, it turned out that the existence theorem could not be established because the famous "difficulty of the small divisors" first noticed by Poincaré turned up unexpectedly. However, the methods developed by Ficken were successfully applied by B. Fleishman in his doctoral thesis to yield the existence of a periodic solution in the same problem when a viscous damping term with a sufficiently large coefficient is included in the differential equation. J. Keller has recently treated the problem of the bowing of a violin string--which belongs in the same general category of nonlinear problems--by quite different methods which yield explicit solutions.

G. B. Whitham

Explosions: The propagation of spherical blast

When the disturbance produced in the surrounding air by the explosion is weak (whether this applies to the whole disturbed region or only beyond a certain distance from the center of the explosion), the propagation may be described by the method used in my paper "The Flow Pattern of a Supersonic Projectile" (Comm. on Pure and App. Math., Vol. V, No. 3, Aug. 1952). The appropriate details were worked out.

With this theory for the behavior at some distance and "Taylor type" solutions near the center of the explosion, one of the remaining questions is how the gap should be bridged. Brinkley and Kirkwood (Phys. Rev. 71, p. 606) suggest an answer which I now believe is reasonable. However, assumptions are made and many people are not satisfied with that work. The problem is difficult and I tried to simplify it by considering the case of the explosion in water. Then, for example, the particle velocity may be neglected, in certain places, in comparison with the changes in the velocity of sound, and the characteristics are approximately lines t = constant. I have had some success in solving the resulting equations, but cannot say yet whether the problem will be solved.

Transonic flow past bodies

I have participated in the seminar on this subject, giving a few lectures on some of the work that has been done in this field. From reading the literature and from various discussions, I felt that some quite new (and if necessary drastic) method of approximation would be valuable. I have made one or two attempts in this direction, but without result so far.

Dynamics of meteorological fronts

I suggested a theory covering one phase of the motion: the development of the cold front towards breaking like a compression wave in gas dynamics, and the "smoothing out" of the warm front as in an expansion wave. The breaking wave may, in general, be replaced by a bore, but in one important region this is impossible since there could be no mass flux through the wave. However, simple assumptions, e.g. that the breaking wave will, in such regions, move with a velocity which is the average of the particle velocities just at breaking, produce realistic effects such as the occlusion process. I am now looking into this part of the problem more carefully.

Breakup of fluid into drops

In connection with the contract which includes this topic, I formulated the problem of a spherical shell of liquid surrounding an expanding sphere of gas. (This is a crude representation of a liquid bomb.) I investigated how the instability of the liquid surface is connected with the motion of the gas sphere in order to find optimum conditions for the breakup of the liquid and to estimate size of drop, etc. I find, however, that Keller and Kolodner had tackled this and similar problems about the same time. I understand that their work includes mine and so I have discontinued it.

Propagation of spherical blast in a star

In its undisturbed state, the star is assumed to be a sphere of gas with internal pressure distribution balancing the forces of gravity. The outward propagation of weak spherically symmetrical disturbances arising in the interior of the star is then investigated. First, a solution of the linearized equation of the motion in the form of an expansion, valid at large distances and also near the head of the

wave, is obtained. Secondly, this solution is corrected by the technique described in the Communications, Vol. V, pp. 301-348, to give the nonlinear theory. Finally the motion of the shocks is determined. The significance of the results is still under consideration; however, one important result is that a shock can increase in strength, instead of decaying, as it would without the effects of gravity. It is hoped that the method will be of more general application and may, for example, be used in the problem of the motion of waves and bores in shallow water of non-uniform depth.

PUBLICATIONS

"Communications on Pure and Applied Mathematics":

Friedman, Bernard, "Propagation in a Non-homogeneous Atmosphere", Vol. IV, No. 2/3, August 1951.

Friedrichs, K. O., "Mathematical Aspects of the Quantum Theory of Fields, Parts I and II", Vol. IV, No. 2/3, August 1951.

Kline, Morris, "An Asymptotic Solution of Maxwell's Equations", Vol. IV, No. 2/3, August 1951.

Marcuvitz, Nathan, "Field Representations in Spherically Stratified Regions", Vol. IV, No. 2/3, August 1951.

Rice, Stephen O., "Reflection of Electromagnetic Waves from Slightly Rough Surfaces", Vol. IV, No. 2/3, August 1951.

Bremmer, H., "The Jumps of Discontinuous Solutions of the Wave Equation", Vol. IV, No. 4, November 1951.

Copson, E. T., "The Transport of Discontinuities in an Electromagnetic Field", Vol. IV, No. 4, November 1951.

Ficken, F. A., "The Continuation Method for Functional Equations", Vol. IV, No. 4, November 1951.

Lax, Peter D., "A Remark on the Method of Orthogonal Projections", Vol. IV, No. 4, November 1951.

Shiffman, M. and Spencer, D. C., "The Force of Impact on a Cone Striking a Water Surface (Vertical Entry)", Vol. IV, No. 4, November 1951.

Chen, Yu Why, "Supersonic Flow through Nozzles with Rotational Symmetry", Vol. V, No. 1, February 1952.

Friedrichs, K. O., "Mathematical Aspects of the Quantum Theory of Fields, Part III", Vol. V, No. 1, February 1952.

Peters, Arthur S., "Water Waves over Sloping Beaches and the Solution of a Mixed Boundary Value Problem for $\Delta^2\phi - k^2\phi = 0$ in a Sector", Vol. V, No. 1, February 1952.

Douglis, Avron, "Some Existence Theorems for Hyperbolic Systems of Partial Differential Equations in Two Independent Variables", Vol. V, No. 2, May 1952.

John, Fritz, "On Integration of Parabolic Equations by Difference Methods", Vol. V, No. 2, May 1952

Lighthill, M. J., "On the Squirming Motion of Nearly Spherical Deformable Bodies through Liquids at Very Small Reynolds Numbers", Vol. V, No. 2, May 1952.

Courant, R., E. Isaacson and M. Rees, "On the Solution of Non-linear Hyperbolic Differential Equations by Finite Differences", Vol. V, No. 3, August 1952.

Grad, Harold, "The Profile of a Steady Plane Shock Wave," Vol. V, No. 3, August 1952.

van Heijenoort, John, "On Locally Convex Manifolds", Vol. V, No. 3, August 1952.

Tricomi, Francesco G., "A New Entire Function Related to a Well-Known Noncontinuable Power Series", Vol. V, No. 3, August 1952.

Whitham, G. B., "The Flow Pattern of a Supersonic Projectile", also Appendix: "Two-Dimensional Steady and One-Dimensional Unsteady Flows", Vol. V, No. 3, August 1952.

Friedrichs, K. O., "Mathematical Aspects of the Quantum Theory of Fields, Part IV: Occupation Number Representation and Fields of Different Kinds", Vol. V, No. 4, December 1952.

Grad, Harold, "Statistical Mechanics, Thermodynamics, and Fluid Dynamics of Systems with an Arbitrary Number of Integrals", Vol. V, No. 4, December 1952.

Lewy, Hans, "On Steady Free Surface Flow in a Gravity Field", Vol. V, No. 4, December 1952.

Montroll, E. W., "Markoff Chains, Wiener Integrals, and Quantum Theory", Vol. V, No. 4, December 1952.

*<u>IMM-NYU Memoranda</u>:

No. 175, "A Note on the Application of Schwinger's Variational Principle to Dirac's Equation of the Electron", by H. Moses, June 1951.

No. 176, "On the Free Energy of a Mixture of Ions: An Extension of Kramer's Theory", by E. W. Montroll and T. H. Berlin, September 1951.

No. 179, "Electrons and Positrons in a Time-Dependent Electromagnetic Field: A Solution in the Schrödinger Picture", by H. Moses, March 1952.

No. 180, "Underwater Explosion of Atomic Bombs - Summary", by Joseph B. Keller, June 1952.

No. 182, "Instability of Liquid Surfaces and the Formation of Drops", by J. B. Keller and I. Kolodner, June 1952.

No. 183, "Decay of Drops by Evaporation and Growth by Condensation", by J. B. Keller, I. Kolodner and P. Ritger, July 1952.

No. 184, "A Study of the Effects of Viscosity and Heat Conductivity on the Transmission of Sound Waves in a Compressible Fluid", by John R. Knudsen, and "Decay of Spherical Sound Pulses due to Viscosity and Heat Conductivity", by Joseph B. Keller, August 1952.

No. 185, "Spherical, Cylindrical and One-Dimensional Flows of Compressible Fluids", by Joseph B. Keller, September 1952.

No. 186, "Thin Unsteady Heavy Jets", by J. B. Keller and M. Weitz, December 1952.

No. 187, "The Propagation of Weak Spherical Shocks in Stars", by G. B. Whitham, December 1952.

No. 188, "Geometrical Acoustics I: The Theory of Weak Shock Waves", by J. B. Keller, January 1953.

No. 189, "Perturbation about Strong Spherical Shock Waves", by C. S. Morawetz, January 1953.

No. 190, "Spherical Waves and Shocks", by G. B. Whitham, February 1953.

*We have listed only the unclassified reports.

Ph.D. THESES

June 1951:

Arzt, Sholom, "On a Mean Value Theorem for Certain Divisor Functions Taken over Exponential Sequences"

Bernardi, Salvatore, "On the Coefficients of Schlicht Functions"

Blank, Albert, "Diffraction and Reflection of Pulses by Wedges and Corners"

Brock, Paul, "The Nature of Solutions of a Rayleigh Type Forced Vibration Equation with a Large Coefficient of Damping"

Cooperman, Philip, "An Extension of the Method of Trefftz for Finding Local Bounds on the Solution and Their Derivations of Boundary Value Problems"

Evans, George, "The Motion of the Interface in Heat Conduction Problems Involving Recrystallization"

Lurye, Jerome, "Electromagnetic Reflection and Transmission Matrices of a Continuously Stratified Anisotropic Medium by Variational Technique"

Milkman, Joseph, "Hermite Polynomials, Hermite Functionals and Their Integrals in Real Hilbert Space"

Pohlo, Frederick, "The Lagrangian Equations of Hydrodynamics: Solutions Which Are Analytic Functions of the Time"

February 1952:

Hirsch, Warren, "On the Maximum Cumulative Sum of Independent Random Variables"

Kruskal, Martin, "The Bridge Theorem for Minimal Surfaces"

Yachter, Morris, "On the Existence of Periodic Solutions of the Differential Equation: $\ddot{x} + F(x) = \varepsilon F(x,\dot{x},T,\varepsilon)$ "

June 1952:

Fleishman, Bernard, "On the Periodic Solutions to an Initial-Value Problem for a Duffing-Type Non-Linear Wave Equation"

February 1953:

Altshuler, Bernard, "Nonlinear Buckling of Spherical Shell"

Bazer, Jack, "Propagation of Plane Electromagnetic Waves Past a Shoreline"

Finkelstein, Abraham, "The Initial Value Problem for Transient Water Waves"

Forman, William, "Abstract Prime Number Theorems"

Kay, Irvin, "A Direct Method for Pulse Wave Diffraction Problems"

Nemerever, Leon, "On the Convergence of Finite Difference Approximations to the Solution of Systems of Quasilinear Hyperbolic Partial Differential Equations with Singular Coefficients"

Rose, Milton, "On the Solution of Hyperbolic Equations by Difference Methods"

Rubin, Hanan, "The Dock of Finite Extent"

Weitz, Mortimer, "Reflection of Water Waves"

Yanowitch, Michael, "Non-Linear Buckling of Circular Elastic Plates"

LECTURE NOTES

1952-53	Friedrichs: Special Topics in Fluid Dynamics
1952-53	Lax: Functional Analysis and Applications
1952-53	Magnus: Discrete Groups
1952-53	Friedman: Theory of Functions of a Real Variable
1952-53	John: Partial Differential Equations
1952-53	Courant: Supplementary Notes--Methods of Mathematical Physics
1952	Shapiro: Lectures on the Theory of Numbers
1952	Douglis: Supplementary Notes--Partial Differential Equations
1952	John: Special Topics in Partial Differential Equations
1952	Courant: Introduction to Mathematical Analysis
1951	Friedrichs: Theory of Wave Propagation
1951	Friedman: Mathematics of Electromagnetic Propagation
1951	Friedrichs: A Chapter in the Theory of Linear Operators in Hilbert Space
1951	Rademacher: A Survey of Higher Mathematics
1951	Courant-Friedrichs: Seminar in Mathematical Physics
1951-52	Bers: Seminar on Mathematical Aspects of Subsonic and Transonic Dynamics